Gravitational Entropy in Cosmological Models

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We discuss whether an appropriately defined dimensionless scalar function might be an acceptable candidate for the gravitational entropy, by explicitly considering Szekeres and Bianchi type VI_h models that admit an isotropic singularity. We also briefly discuss other possible gravitational entropy functions, including an appropriate measure of the velocity dependent Bel-Robinson tensor.

1. INTRODUCTION

Penrose (1979) has argued that the initial cosmological singularity must be one of low entropy in order to explain the high isotropy of the observed universe and to be consistent with the second law of thermodynamics. Since the matter was presumably in thermal equilibrium at the initial singularity, this implies low entropy in the gravitational field. Penrose has also conjectured that such a gravitational entropy should be related to a suitable measure of the Weyl curvature. The search for a suitable candidate for the "gravitational entropy" is therefore of current interest, particularly in the approach to the initial cosmological singularity (Tod, 1990; Goode *et al*., 1992). In the quiescent cosmological paradigm (Barrow, 1978) the universe began in a highly regular state and subsequently evolved towards irregularity.

Penrose (1979) originally proposed that the the Weyl tensor is zero at the big bang singularity, implying the subsequent evolution is close to a Friedmann–Robertson–Walker (FRW) model. However, this requirement is too strong. For example, in perfect fluid spacetimes Anguige and Tod (1999) have proven uniqueness results that show that if the Weyl tensor is zero at the big bang, then the

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spacetime geometry must be exactly FRW in a neighbourhood of the big-bang. This has motivated the idea that some appropriate dimensionless scalar is asymptotically zero. The search for a gravitational entropy then reduces to a search for this scalar function. Quiescent cosmology and the ideas of Penrose provide the motivation of the definition of an isotropic singularity (Goode and Wainwright, 1985); essentially a spacetime admits an isotropic singularity if the 'physical' spacetime is conformally related to an 'unphysical' spacetime such that there exists a time function T with the property that at $T = 0$ the conformal factor vanishes (corresponding to the cosmological singularity) but that the conformally related metric is regular. It was proven in Goode and Wainwright (1985) that in the class of models with an isotropic singularity $P \to 0$ as $T \to 0$, where P is the ratio of the Weyl curvature squared to the Ricci curvature squared (see Eq. (3) below).

Recently, Lake and Pelavas (LP) (2000) considered a class of "gravitational epoch" functions that are a dimensionless scalar field constructed from the Riemann tensor and its covariant derivatives only (such as, for example, P). They discussed whether such functions can act as a "gravitational entropy" by determining whether it is monotone along a suitable set of (smooth) timelike trajectories. In particular, LP considered the set of homothetic trajectories of a self-similar spacetime (since such spacetimes are believed to play an important role in describing the asymptotic properties of more general models (Carr and Coley, 1999)). They showed that the Lie derivative of any "dimensionless" scalar along a homothetic vector field (HVF) is zero, and concluded that such functions are not acceptable candidates for the gravitational entropy. They suggested considering other options for a "gravitational epoch" function. Other dimensionless scalars constructed from the Riemann tensor and its covariant derivatives have been considered (see, for example, Goode *et al.*, 1992). Other alternatives, such as those involving the Bel-Robinson tensor, were suggested in Pelavas and Lake (1998).

In this paper, by explicitly considering classes of Szekeres and Bianchi VI$_h$ models that admit an isotropic singularity, we revisit the conclusion of LP that P (for example) is not an acceptable candidate for the gravitational entropy. First, we take the view, unlike that taken in LP, that a purely gravitational entropy selects spacetimes as being of cosmological interest according to a thermodynamic principle. For example, in General Relativity (GR) there exist solutions for which the physical energy density is negative; however, we do not disregard GR or the notion of energy density within GR – rather, we use the criterion of negative energy density to characterize those solutions which are not physical. Second, homothetically self-similar spacetimes represent asymptotic equilibrium states (since they describe the asymptotic properties of more general models (Carr and Coley, 1999)), and the LP result is perhaps consistent with this interpretation since the entropy does not change in these equilibrium models, and perhaps consequently supports the idea that P (for example) represents a "gravitational entropy." Therefore, we will investigate the behavior of P asymptotically as the

self-similar solution is approached in cosmological models. We also consider various other options for a purely "gravitational entropy," including an appropriate measure of the Bel-Robinson tensor.

2. SZEKERES MODEL

We consider the class II Szekeres solutions, which are spatially inhomogeneous models with irrotational dust as a source. A comprehensive analysis of the Szekeres models can be found in Goode and Wainwright (1982a). In the notation of Goode and Wainwright we set $k = 0 = \beta_-$, the line-element in comoving coordinates has the form

$$ds^2 = T^4[-dT^2 + dx^2 + dy^2 + (A - \beta_+ T^2)^2 dz^2] \tag{1}$$

where

$$A = a(z) + b(z)x + c(z)y - 5\beta_+(z)(x^2 + y^2). \tag{2}$$

The fluid 4-velocity is $u = T^{-2}\frac{\partial}{\partial T}$ and the energy density is $\mu = 12T^{-6}[1 - (\beta_+/A)T^2]^{-1}$. These cosmological models admit an isotropic singularity at $T = 0$; moreover, it has been shown (Goode and Wainwright, 1982b) that the general Szekeres class with $\beta_- = 0$ also admits an isotropic singularity. If $\beta_+ = 0$ in (1), then we obtain the associated flat FRW dust solution.

The standard gravitational epoch function, P, for (1) is

$$P \equiv \frac{C_{abcd}C^{abcd}}{R_{ab}R^{ab}} = \frac{4}{3}\frac{T^4\beta_+^2}{A^2}. \tag{3}$$

This has been shown (Goode *et al.*, 1992) to behave appropriately in these models; i.e., P is monotonically increasing away from the isotropic singularity. Perhaps an alternative choice is to use the Bel-Robinson tensor (Pelavas and Lake, 1998) to construct a velocity dependent gravitational epoch function. Using the fluid 4-velocity, we construct the positive scalar

$$W = T_{abcd}u^a u^b u^c u^d = \frac{24\beta_+^2}{T^8(\beta_+ T^2 - A)^2} \tag{4}$$

which has the same units as $C_{abcd}C^{abcd}$, and hence to obtain a dimensionless scalar we normalize by the square of $\mu = T_{ab}u^a u^b$; i.e.,

$$\widetilde{P} = \frac{W}{\mu^2} = \frac{T^4\beta_+^2}{6A^2}. \tag{5}$$

Noting that in this case we have $\widetilde{P} = P/8$, then $\widetilde{P}$ also behaves appropriately as the isotropic singularity is approached. This relationship is a consequence of the following two facts. First, the magnetic part of Weyl is zero, $H_{ab} = 0$,

therefore $C_{abcd}C^{abcd}$ is equivalent to W modulo a positive constant. Second, the Ricci invariant $R_{ab}R^{ab} = R^2$ and for dust $R = \mu$. In these models, the choices are limited for constructing dimensionless ratios of zeroth order invariants since all Carminati-McLenaghan (CM) invariants[3] can be expressed in terms of the Ricci scalar R and $Re(w_1) \sim C_{abcd}C^{abcd}$,

$$R = \frac{12A}{T^6(A - \beta_+ T^2)}, \quad w_1 = \frac{24\beta_+^2}{T^8(A - \beta_+ T^2)^2} \tag{6}$$

with syzygies for the Ricci invariants,

$$r_1 = \frac{3}{16}R^2, \quad r_2 = \frac{3}{64}R^3, \quad r_3 = \frac{21}{1024}R^4. \tag{7}$$

Since these spacetimes are Petrov type D, the Weyl invariants satisfy $6w_2^2 = w_1^3$. The mixed invariants give syzygies

$$m_1 = 0, \quad m_2 = \frac{1}{16}w_1 R^2 = m_3, \quad m_4 = -\frac{1}{128}w_1 R^3, \quad m_5 = \frac{1}{16}w_2 R^2. \tag{8}$$

To study the Lie derivative of P (or $\widetilde{P}$) along a HVF we first recall a well known result (Carr and Coley, 1999). Any FRW model with an equation of state $p = -\mu/3$ admits a timelike HVF. In addition, only the flat FRW models with an equation of state $p = \alpha\mu$ and power law dependence on the scale function admit a HVF. If β_+ is set to zero in (1), then the resulting flat FRW model is in non-standard coordinates; although there exists a HVF, finding it in this coordinate system is difficult due to the presence of the functions a, b and c. Determining a coordinate transformation into more familiar coordinates, where HVF's can easily be found, is also difficult. We use the following simplifying assumptions in (1), $a = 1, b = 0, c = 0$, and consider deviations from flat FRW by redefining $\beta_+ \to \epsilon\beta_+$ for small ϵ. It can be shown that

$$\xi = \frac{\phi}{3}\left(T\frac{\partial}{\partial T} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) \tag{9}$$

is a HVF in the associated flat FRW model and deviates[4] from homotheticity in (1) to first order in ϵ. Since $\lim_{T\to 0}(\text{Ł}_\xi g_{ab} - 2\phi g_{ab}) = 0$ we find that (9) becomes a HVF as the isotropic singularity is approached. In this limit then $\text{Ł}_\xi P = 0$, as expected (Pelavas and Lake, 2000), and so P approaches a constant. This type of behavior is desirable if we would like to interpret P, in some sense, as a gravitational entropy possessing a critical point at early times in the evolution. Moreover, requiring that P be monotonically increasing at early times along a timelike ξ ($\text{Ł}_\xi P > 0$) places restrictions on β_+.

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[3] We note that in this case any complex CM invariants have vanishing imaginary part.
[4] As a special case, if $\beta_+ = C/z^2$ then (9) is also a HVF of (1); therefore, as a consequence of Pelavas and Lake (2000), $\text{Ł}_\xi P = 0$.

Since

$$\unicode{x141}_\xi P = \frac{8}{9}\frac{\phi\epsilon^2 T^4 \beta_+(2\beta_+ + z\beta'_+)}{(1 - \epsilon\beta_+ R^2)^3} = \frac{8}{9}\phi T^4 \beta_+(2\beta_+ + z\beta'_+)\epsilon^2 + O(\epsilon^3) \quad (10)$$

and

$$\xi \cdot \xi = \left(\frac{\phi}{3}\right)^2 T^4 \left\{-T^2 + \frac{R^2}{5} + [1 - \epsilon\beta_+(R^2 + T^2)]^2 z^2\right\} \quad (11)$$

where $R^2 \equiv 5(x^2 + y^2)$, then as $T \to 0$, ξ remains timelike if $R = 0$ and $z \to 0$ subject to the requirement that $\lim_{z\to 0} z\beta_+$ be bounded. Assuming $z\beta_+$ is analytic near $z = 0$ gives the form

$$\beta_+ = \frac{b_0}{z} + b_1 + b_2 z + \cdots \quad (12)$$

To leading order in ϵ we factor β_+ from (10) and use (12) to obtain $\lim_{z\to 0}(2 + z\beta'_+/\beta_+) = 1$; thus $\unicode{x141}_\xi P > 0$ along a timelike ξ.

3. BIANCHI VI$_H$ MODEL

It has been shown (Wainwright and Anderson, 1984) for the Bianchi VI$_h$ class that a choice of parameters can result in the quasi-isotropic stage beginning at the initial singularity, giving rise to an isotropic singularity for these spacetimes. In the notation of Wainwright and Anderson (1984), we set $\alpha_s = 0$ and $\alpha_m = 1$, so that the line-element in conformal time coordinates is

$$ds^2 = \tau^{4/(3\gamma-2)}\left(-A^{2(\gamma-1)}d\tau^2 + A^{2q_1}dx^2 + A^{2q_2}e^{2r[s+(3\gamma-2)]x}dy^2\right.$$
$$\left. + A^{2q_3}e^{2r[s-(3\gamma-2)]x}dz^2\right) \quad (13)$$

where

$$A^{2-\gamma} = 1 + \alpha_c\tau^2, \quad q_1 = \frac{\gamma}{2}, \quad q_2 = \frac{2-\gamma+s}{4}, \quad q_3 = \frac{2-\gamma-s}{4},$$
$$s^2 = (3\gamma+2)(2-\gamma), \quad r^2 = \frac{(3\gamma+2)\alpha_c}{4(2-\gamma)(3\gamma-2)^2}. \quad (14)$$

These spacetimes have a perfect fluid source with equation of state $p = (\gamma - 1)\mu$, $1 \leq \gamma < 2$. The fluid 4-velocity is $u = A^{1-\gamma}\tau^{-2/(3\gamma-2)}\frac{\partial}{\partial\tau}$ and the energy density is $\mu = 12A^{-\gamma}(3\gamma - 2)^{-2}\tau^{-6\gamma/(3\gamma-2)}$. Since α_s has been set to zero, then the isotropic singularity occurs at $\tau = 0$. The parameter α_c determines the curvature of the spacelike hypersurfaces orthogonal to u; if $\alpha_c = 0$ we obtain the flat FRW solution. We shall consider deviations about this flat FRW model by assuming α_c is small.

To leading order in α_c, the gravitational epoch function for (13) is

$$P = \frac{4}{3} \frac{\gamma^2 (3\gamma - 2)^2 \tau^4}{(\gamma - 2)^2 [3(\gamma - 1)^2 + 1]} \alpha_c^2 + O(\alpha_c^3), \tag{15}$$

which is positive and $P \to 0$ as $\tau \to 0^+$. Using the Bel-Robinson tensor and the energy density we find

$$\widetilde{P} = \frac{\gamma^2 (3\gamma - 2)^2 \tau^4}{6(\gamma - 2)^2} \alpha_c^2 + O(\alpha_c^3). \tag{16}$$

Consequently to leading order in α_c we have that $\widetilde{P} = [3(\gamma - 1)^2 + 1]P/8$, and again P and $\widetilde{P}$ are directly proportional. Unlike the Szekeres class above, the magnetic part of the Weyl tensor with respect to u does not vanish here unless[5] $\gamma = 4/3$. This relationship between P and $\widetilde{P}$ becomes evident if we consider the expansions for the relevant invariants of the electric and magnetic parts of Weyl

$$E_{ab} E^{ab} = \frac{24\gamma^2}{(\gamma - 2)^2 (3\gamma - 2)^2 \tau^{8/(3\gamma - 2)}} \alpha_c^2 + O(\alpha_c^3),$$

$$\tag{17}$$

$$H_{ab} H^{ab} = \frac{2\gamma^2 (3\gamma + 2)(3\gamma - 4)^2}{(2 - \gamma)^3 (3\gamma - 2)^2 \tau^{6(2-\gamma)/(3\gamma - 2)}} \alpha_c^3 + O(\alpha_c^4).$$

For α_c small, $H_{ab} H^{ab} \sim 0$ and $C_{abcd} C^{abcd} \sim E_{ab} E^{ab} \sim W$, additionally Einstein's equations give $R_{ab} R^{ab} = [3(\gamma - 1)^2 + 1]\mu^2$, the relationship now follows.

As $\alpha_c \to 0$ or if $\tau \to 0^+$, the vector

$$\xi = \frac{\phi}{3} \frac{(3\gamma - 2)}{\gamma} \left(\tau \frac{\partial}{\partial \tau} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \tag{18}$$

gives $(Ł_\xi g_{ab} - 2\phi g_{ab}) \to 0$ separately in both limits; therefore, ξ is a HVF in the flat FRW limit or as the isotropic singularity is approached. Setting $x = y = z = 0$ we obtain for the magnitude of ξ

$$\xi \cdot \xi = - \left(\frac{\phi}{3} \right)^2 \frac{(3\gamma - 2)^2}{\gamma^2} \tau^{6\gamma/(3\gamma - 2)} A^{2(\gamma - 1)}; \tag{19}$$

thus by continuity ξ will be timelike in some neighborhood of $x = y = z = 0$ arbitrarily close to the isotropic singularity. To leading order in α_c, the behavior of P along ξ is given by

$$Ł_\xi P = \frac{16}{9} \frac{\phi\gamma (3\gamma - 2)^3 \tau^4}{(\gamma - 2)^2 [3(\gamma - 1)^2 + 1]} \alpha_c^2 + O(\alpha_c^3) \tag{20}$$

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[5] H_{ab} will also vanish if $\alpha_c = 0$.

which is positive for $1 \leq \gamma < 2$; hence P will be monotonically increasing at early times along the timelike ξ, and as $\tau \to 0^+$ then P will tend to a constant (which in these models is zero).

We now show that the Weyl curvature hypothesis does not necessarily put restrictions on the Petrov type. In this class of spacetimes the Weyl invariants of CM have vanishing imaginary parts, and in general do not always satisfy the syzygy $w_1^3 - 6w_2^2 = 0$; therefore, the metric(13) is almost always Petrov type I. However for $4/3 < \gamma < 2$ there always exists a time τ_* where the Petrov type specializes to either II or D; this is given by

$$\tau_* = \frac{3}{2} \frac{2 - \gamma}{\sqrt{2\alpha_c(3\gamma - 4)}}, \tag{21}$$

otherwise the Petrov type is I. When $\gamma = 4/3$ the syzygy is satisfied and by choosing the aligned Newman-Penrose tetrad

$$\ell = A^{-2/3} \frac{\partial}{\partial \tau} + \frac{\partial}{\partial z}, n = \frac{1}{2t^2} \left(\frac{\partial}{\partial \tau} - A^{2/3} \frac{\partial}{\partial z} \right),$$

$$m = \frac{1}{\sqrt{2}\tau A^{2/3}} \left(\frac{\partial}{\partial x} + ie^{-4rx} \frac{\partial}{\partial y} \right) \tag{22}$$

we find that this is in fact Petrov type D for all $\tau > 0$. It would appear that an intermediate algebraic specialization of the Weyl tensor during the evolution does not affect the increasing anisotropy that is indicated by the gravitational epoch function (15); indeed, these spacetimes begin with small anisotropy close to the isotropic singularity and approach the anisotropic vacuum plane wave metrics at late times.

As is shown in Pelavas and Lake (2000), *any* dimensionless ratio of invariants is constant along a HVF. Therefore, depending on its monotonicity, it may also serve as a gravitational epoch function when the cosmological model admits an asymptotic timelike HVF. Here we illustrate this point by considering a dimensionless ratio of differential invariants; to second order in α_c we have

$$P_1 = \frac{\nabla_a C_{bcde} \nabla^a C^{bcde}}{\nabla_a R_{bc} \nabla^a R^{bc}} = \frac{40}{9} \frac{(3\gamma - 2)^2 \tau^4}{(\gamma - 2)^2 [9(\gamma - 1)^2 + 5]} \alpha_c^2 + O(\alpha_c^3) \tag{23}$$

and

$$Ł_\xi P_1 = \frac{160}{27} \frac{\phi(3\gamma - 2)^3 \tau^4}{\gamma(\gamma - 2)^2 [9(\gamma - 1)^2 + 5]} \alpha_c^2 + O(\alpha_c^3). \tag{24}$$

Clearly as $\tau \to 0^+$, $P_1 \to 0$ and from (24) it is also monotonically increasing for $1 \leq \gamma < 2$. Nevertheless, the invariants of (23) diverge in a similar manner to the invariants of (15); i.e., as $\tau \to 0^+$, $\nabla_a C_{bcde} \nabla^a C^{bcde}$ and $\nabla_a R_{bc} \nabla^a R^{bc} \to -\infty$.

4. DISCUSSION

We have considered a class of spatially inhomogeneous Szekeres solutions with the line-element (1). We show that there exists a HVF (9) as the isotropic singularity is approached, and show that $Ł_\xi P = 0$ in this limit, so that P approaches a constant as expected (Pelavas and Lake, 2000). Moreover, to leading order we show that $Ł_\xi P > 0$ along an approach to the singularity in which the HVF ξ remains timelike; i.e., P is monotonically increasing at early times along ξ. We then considered a class of Bianchi VI$_h$ models with an isotropic singularity with line-element (13), parameterized by α_c (which measures deviations about the flat FRW model). Assuming α_c is small, to leading order we display a vector ξ which is a HVF in the flat FRW limit, and show that the gravitational epoch function $P \to 0$ as the isotropic singularity is approached and that P is monotonically increasing at early times along timelike ξ.

Therefore, in the isotropic singularity cosmological models we have studied we have found that $P \to 0$ asymptotically as the self-similar cosmological model is approached, in support of the idea that these homothetically self-similar spacetimes represent asymptotic equilibrium states. Moreover, we have provided evidence that P is monotonically increasing as the models evolve away from these equilibrium states, which perhaps lends support to the idea that P represents a "gravitational entropy."

We also found that for both the Szekeres models and the Bianchi VI$_h$ models (to leading order in α_c), the standard gravitational epoch function P, and the normalized Bel-Robinson epoch function $\widetilde{P}$, are proportional (and hence equivalent as gravitational epoch functions). The question remains as to whether this will be true for all models with an isotropic singularity. In general, for a perfect fluid source we have that $\widetilde{P} \sim (E^2 + H^2)/\mu^2$ and $P \sim (E^2 - H^2)/(\mu^2 + 3p^2)$. Assuming an equation of state of the form $p = \alpha\mu$ gives $P \sim (E^2 - H^2)/[(1 + 3\alpha^2)\mu^2]$. Clearly if H^2 is negligible with respect to E^2, then P and $\widetilde{P}$ will be effectively proportional. It may also be of interest to consider cosmological models where P and $\widetilde{P}$ differ. In particular, whenever the Petrov type is III, N or O then all zeroth order Weyl invariants vanish, and hence P vanishes but $\widetilde{P}$ does not necessarily vanish. An example of such cosmological models are the Oleson (2003) solutions, which are Petrov type N with a perfect fluid source. In these models P vanishes but $\widetilde{P}$ does not; it is of interest to determine if these models can admit an isotropic singularity. We may also consider the possibility that the denominator of P vanishes. It is known that all zeroth order Ricci invariants vanish in spacetimes of Plebański-Petrov (PP)-type N or O (Stephani *et al.*, 2003), although PP-N type spacetimes are unphysical since they cannot satisfy the weak energy conditions. Such spacetimes might be regarded as late time asymptotic states of some cosmological models, and in this case a dominance of the Weyl invariant over the Ricci invariant would likely lead to a monotonically increasing P asymptotically.

A classic problem in cosmology is finding a way to explain the very high degree of isotropy observed in the cosmic microwave background. In GR cosmological models admitting an isotropic singularity are of zero measure, so that isotropy is a special rather than generic feature of cosmological models. Hence, a dynamical mechanism which is able to produce isotropy, such as inflation, is needed. However, inflation requires sufficiently homogeneous initial data in order to begin (Kolb and Turner, 1990); hence the isotropy problem remains open to debate in standard cosmology. Recently, it has been argued that an isotropic singularity is typical in brane world cosmological models (Coley, 2002a,b, 2004). Hence brane cosmology would have the very attractive feature that it provides for the necessary sufficiently smooth initial conditions which might, in turn, be consistent with entropy arguments and the second law of thermodynamics.

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